A derivation of Dirac equation from standing waves

1st/Dec/2013

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I consider about a situation that three of A standing wave solution of non-linear wave equation are exist and in the center of the solutions these standing waves are formed approximately as follows.

$$\varphi_1 = \left[0, 1, \frac{\sqrt{2}}{2} i, -\frac{\sqrt{2}}{2}, 0, 1 \right] \begin{bmatrix} \cos\left(\frac{\omega_0 x}{c}\right)\frac{\vec{x}}{|x|} \\ \sin\left(\frac{\omega_0 x}{c}\right)\frac{\vec{x}}{|x|} \\ \cos\left(\frac{\omega_0 y}{c}\right)\frac{\vec{y}}{|y|} \\ \sin\left(\frac{\omega_0 y}{c}\right)\frac{\vec{y}}{|y|} \\ \cos\left(\frac{\omega_0 z}{c}\right)\frac{\vec{z}}{|z|} \\ \sin\left(\frac{\omega_0 z}{c}\right)\frac{\vec{z}}{|z|} \end{bmatrix} \exp(-i\omega t)$$

$$\varphi_2 = \left[-i, 0, \frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2}, 1, 0 \right] \begin{bmatrix} \cos\left(\frac{\omega_0 x}{c}\right)\frac{\vec{x}}{|x|} \\ \sin\left(\frac{\omega_0 x}{c}\right)\frac{\vec{x}}{|x|} \\ \cos\left(\frac{\omega_0 y}{c}\right)\frac{\vec{y}}{|y|} \\ \sin\left(\frac{\omega_0 y}{c}\right)\frac{\vec{y}}{|y|} \\ \cos\left(\frac{\omega_0 z}{c}\right)\frac{\vec{z}}{|z|} \\ \sin\left(\frac{\omega_0 z}{c}\right)\frac{\vec{z}}{|z|} \end{bmatrix} \exp(-i\omega t)$$

Where ω_0 is angular frequency $\frac{\vec{x}}{|x|}$、$\frac{\vec{y}}{|y|}$、$\frac{\vec{z}}{|z|}$ are unit vectors of x, y, z.

I seek the real part of the standing waves.

$$\mathrm{Re}(\varphi_1) = \mathrm{Re}\left(\left[\sin\left(\frac{\omega_0 x}{c}\right) \quad , \quad \left(\frac{\sqrt{2}}{2}i\right)\cos\left(\frac{\omega_0 y}{c}\right) + \left(-\frac{\sqrt{2}}{2}\right)\sin\left(\frac{\omega_0 y}{c}\right) \quad , \quad \sin\left(\frac{\omega_0 z}{c}\right)\right]\exp(-i\omega_0 t)\right)$$

$$= \left[\cos(\omega_0 t)\sin\left(\frac{\omega_0 x}{c}\right), \left(\frac{\sqrt{2}}{2}\right)\sin(\omega_0 t)\cos\left(\frac{\omega_0 y}{c}\right) + \left(-\frac{\sqrt{2}}{2}\right)\cos(\omega_0 t)\sin\left(\frac{\omega_0 y}{c}\right), \cos(\omega_0 t)\sin\left(\frac{\omega_0 z}{c}\right)\right]$$

$$= \left[\cos(\omega_0 t)\sin\left(\frac{\omega_0 x}{c}\right) \quad , \quad \frac{\sqrt{2}}{2}\sin\left(\omega_0 t - \frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega_0 t)\sin\left(\frac{\omega_0 z}{c}\right)\right]$$

$$\mathrm{Re}(\varphi_2) = \left[\sin(\omega_0 t)\cos\left(\frac{\omega_0 x}{c}\right) \quad , \quad \frac{\sqrt{2}}{2}\sin\left(\omega_0 t + \frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega_0 t)\cos\left(\frac{\omega_0 z}{c}\right)\right]$$

I found that they are combinations of a couple of standing waves and one of backward wave or traveling wave.
And if we make a product $\varphi_1 \varphi_2$ with spin it is also a combination of a couple of standing waves and one of backward wave or traveling wave as follows.

$$\mathrm{Re}\left([\varphi_1 \varphi_2]\begin{bmatrix}1\\0\end{bmatrix}\right) = \left[\cos(\omega_0 t)\sin\left(\frac{\omega_0 x}{c}\right) \quad , \quad \frac{\sqrt{2}}{2}\sin\left(\omega_0 t - \frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega t)\sin\left(\frac{\omega_0 z}{c}\right)\right]$$

$$\mathrm{Re}\left([\varphi_1 \varphi_2]\begin{bmatrix}0\\1\end{bmatrix}\right) = \left[\sin(\omega_0 t)\cos\left(\frac{\omega_0 x}{c}\right) \quad , \quad \frac{\sqrt{2}}{2}\sin\left(\omega_0 t + \frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega_0 t)\cos\left(\frac{\omega_0 z}{c}\right)\right]$$

$$\mathrm{Re}\left([\varphi_1 \varphi_2]\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}\right) = \left[\frac{\sqrt{2}}{2}\sin\left(\omega_0 t + \frac{\omega_0 x}{c}\right) \quad , \quad \sin(\omega_0 t)\cos\left(\frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega_0 t)\sin\left(\frac{\omega_0 z}{c} + \frac{\pi}{4}\right)\right]$$

$$\mathrm{Re}\left([\varphi_1 \varphi_2]\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{bmatrix}\right) = \left[-\frac{\sqrt{2}}{2}\sin\left(\omega_0 t - \frac{\omega_0 x}{c}\right) \quad , \quad -\cos(\omega_0 t)\sin\left(\frac{\omega_0 y}{c}\right) \quad , \quad \cos(\omega_0 t)\sin\left(\frac{\omega_0 z}{c} - \frac{\pi}{4}\right)\right]$$

$$\mathrm{Re}\left([\varphi_1\ \varphi_2]\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}i\end{bmatrix}\right)=\left[\cos(\omega_0 t)\sin\left(\frac{\omega_0 x}{c}+\frac{\pi}{4}\right),\ \sin\left(\omega_0 t+\frac{\pi}{4}\right)\cos\left(\frac{\omega_0 y}{c}-\frac{\pi}{4}\right),\ -\frac{\sqrt{2}}{2}\sin\left(\omega_0 t-\frac{\omega_0 z}{c}\right)\right]$$

$$\mathrm{Re}\left([\varphi_1\ \varphi_2]\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}i\end{bmatrix}\right)=\left[\cos(\omega_0 t)\sin\left(\frac{\omega_0 x}{c}-\frac{\pi}{4}\right),-\sin\left(\omega_0 t+\frac{\pi}{4}\right)\cos\left(\frac{\omega_0 y}{c}+\frac{\pi}{4}\right),\ \frac{\sqrt{2}}{2}\sin\left(\omega_0 t+\frac{\omega_0 z}{c}\right)\right]$$

I assume that these standing waves are traveling with speed $\vec{v}$.

Where c is the light speed.

When we observe the time and distance from a static system that travel with speed $\vec{v}$ then

$$t'=\gamma\left(t-\frac{\vec{v}\cdot\vec{x}}{c^2}\right)$$
$$x'=\gamma(x-v_x t)$$
$$y'=\gamma(y-v_y t)$$
$$z'=\gamma(z-v_z t)$$

Where $\gamma=\dfrac{1}{\sqrt{1-\frac{|\vec{v}|^2}{c^2}}}$

In the static system the time and distances are replaced as

$t \to t'$

$x \to x'$

$y \to y'$

$z \to z'$

$$\varphi_1 = \left[0\,,1\,,\frac{\sqrt{2}}{2}i\,,-\frac{\sqrt{2}}{2}\,,0\,,1 \right] \begin{bmatrix} \cos\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \sin\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \cos\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \sin\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \cos\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \\ \sin\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \end{bmatrix} \exp\left(-i\omega_0 t'\right)$$

$$\varphi_2 = \left[-i\,,0\,,\frac{\sqrt{2}}{2}i\,,\frac{\sqrt{2}}{2}\,,1\,,0 \right] \begin{bmatrix} \cos\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \sin\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \cos\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \sin\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \cos\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \\ \sin\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \end{bmatrix} \exp\left(-i\omega_0 t'\right)$$

I assume a_1, a_2 as follows

$$a_1 = [a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}] = \left[0, 1, \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2}, 0, 1\right]$$

$$a_2 = [a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}] = \left[-i, 0, \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}, 1, 0\right]$$

Then the inner products of a_1, a_2 are

$$a_1^* \bullet a_1 = 3$$
$$a_1^* \bullet a_2 = 0$$
$$a_2^* \bullet a_2 = 3$$
$$a_2^* \bullet a_1 = 0$$

$$a_n^* \bullet a_m = 3\delta_{nm}$$

Where δ_{nm} is the Kronecker delta.

$$\delta_{nm} = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases} \qquad n = 1,2 \qquad m = 1,2$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \sin\left(\frac{\omega_0 x'}{c}\right)\frac{\vec{x}'}{|x'|} \\ \cos\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \sin\left(\frac{\omega_0 y'}{c}\right)\frac{\vec{y}'}{|y'|} \\ \cos\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \\ \sin\left(\frac{\omega_0 z'}{c}\right)\frac{\vec{z}'}{|z|} \end{bmatrix}$$

then

$$\frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l} b_n{}^T b_m dx'dy'dz' = \frac{1}{2}\delta_{nm}$$
$$\varphi_1 = a_1 b \exp(-i\omega_0 t')$$
$$\varphi_2 = a_2 b \exp(-i\omega_0 t')$$

I assume that

$$l >> \frac{c}{f}$$

Where l is the resolution of distance.

then

$$\frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l} b^T a_n^{*T} a_m b dx'dy'dz'$$

$$= \frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l}\left[b_1, b_2, b_3, b_4, b_5, b_6\right]\begin{bmatrix} a_{n1}^* \\ a_{n2}^* \\ a_{n3}^* \\ a_{n4}^* \\ a_{n5}^* \\ a_{n6}^* \end{bmatrix}\left[a_{m1}, a_{m2}, a_{m3}, a_{m4}, a_{m5}, a_{m6}\right]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} dx'dy'dz'$$

$$= \frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l}\left[b_1, b_2, b_3, b_4, b_5, b_6\right]\begin{bmatrix} a_{n1}^* a_{m1}, a_{n1}^* a_{m2}, \ldots\ldots a_{n1}^* a_{m6} \\ a_{n2}^* a_{m1}, a_{n2}^* a_{m2}, \ldots\ldots a_{n2}^* a_{m6} \\ . \\ . \\ a_{n6}^* a_{m1}, a_{n6}^* a_{m2}, \ldots\ldots a_{n6}^* a_{m6} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} dx'dy'dz'$$

$$= \frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l}\left[b_1, b_2, b_3, b_4, b_5, b_6\right]\begin{bmatrix} a_{n1}^* a_{m1} b_1 + a_{n1}^* a_{m2} b_2 + \ldots\ldots + a_{n1}^* a_{m6} b_6 \\ a_{n2}^* a_{m1} b_1 + a_{n2}^* a_{m2} b_2 + \ldots\ldots + a_{n2}^* a_{m6} b_6 \\ . \\ . \\ a_{n6}^* a_{m1} b_1 + a_{n6}^* a_{m2} b_2 + \ldots\ldots a_{n6}^* a_{m6} b_6 \end{bmatrix} dx'dy'dz'$$

$$= \frac{1}{l^3} \int_{x=0}^{l} \int_{y=0}^{i} \int_{z}^{l} \begin{bmatrix} b_1 a_{n1}^* a_{m1} b_1 + b_1 a_{n1}^* a_{m2} b_2 + \ldots\ldots + b_1 a_{n1}^* a_{m6} b_6 \\ + b_2 a_{n2}^* a_{m1} b_1 + b_2 a_{n2}^* a_{m2} b_2 + \ldots\ldots + b_2 a_{n2}^* a_{m6} b_6 \\ . \\ . \\ + b_6 a_{n6}^* a_{m1} b_1 + b_6 a_{n6}^* a_{m2} b_2 + \ldots\ldots b_6 a_{n6}^* a_{m6} b_6 \end{bmatrix} dx' dy' dz'$$

$$= \frac{1}{l^3} \int_{x=0}^{l} \int_{y=0}^{i} \int_{z}^{l} \begin{bmatrix} b_1 a_{n1}^* a_{m1} b_1 \\ + b_2 a_{n2}^* a_{m2} b_2 \\ . \\ . \\ + b_6 a_{n6}^* a_{m6} b_6 \end{bmatrix} dx' dy' dz' = \frac{1}{2} \left[a_{n1}^* a_{m1} + a_{n2}^* a_{m2} \ldots. + a_{n6}^* a_{m6} \right]$$

$$= \frac{1}{2} a_n^* \bullet a_m$$

$$= \frac{3}{2} \delta_{nm}$$

I assume that

E is the energy m is the weight $\vec{p}$ is the momentum

f_0 is the frequency when it is stationary.

f is the frequency when it is traveling.

From the theory of special relativity.

$$E = mc^2\gamma$$

$$\vec{p} = m\vec{v}\gamma$$

$$f = f_0\gamma$$

h is the constant of plank

From the Quantum mechanics.

$$E = hf = hf_0\gamma$$

Therefore the frequency is proportional to the weight as

$$f_0 = \frac{mc^2}{h}$$

This is consistent with A standing wave solution of non-linear wave equation[A condition when the frequency is in proportion to the weight]

If ω_0 is the angular frequency then

$$\omega_0\gamma = 2\pi f_0\gamma = 2\pi\frac{E}{h} = \frac{E}{\hbar}$$

$$\omega_0\gamma\frac{\vec{v}}{c^2} = 2\pi f_0\gamma\frac{\vec{v}}{c^2} = 2\pi\frac{mc^2}{h}\gamma\frac{\vec{v}}{c^2} = \frac{m\vec{v}\gamma}{\hbar} = \frac{\vec{p}}{\hbar}$$

$$\omega_0 t' = \omega_0\gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right) = \omega_0\gamma t - \omega_0\gamma\frac{\vec{v}\cdot\vec{x}}{c^2} = \frac{Et - \vec{p}\cdot\vec{x}}{\hbar}$$

From the Theory of special relativity the square of E is

$$E^2 = m^2c^4\gamma^2 = m^2c^4\gamma^2 = m^2c^4 \frac{1}{1-\frac{|\vec{v}|^2}{c^2}}$$

$$= m^2c^4\left(\frac{1-\frac{|\vec{v}|^2}{c^2}+\frac{|\vec{v}|^2}{c^2}}{1-\frac{|\vec{v}|^2}{c^2}}\right) = m^2c^4\left(1+\frac{\frac{|\vec{v}|^2}{c^2}}{1-\frac{|\vec{v}|^2}{c^2}}\right)$$

$$= m^2c^4 + m^2c^4\left(\frac{\frac{|\vec{v}|^2}{c^2}}{1-\frac{|\vec{v}|^2}{c^2}}\right) = m^2c^4 + m^2c^2\gamma^2|\vec{v}|^2$$

$$= m^2c^4 + c^2p^2$$

Where $p = |\vec{p}|$

Therefore

$$E^2 = m^2c^4 + c^2p^2$$

I consider about standing wave φ as following

$$\varphi = \begin{bmatrix} \omega_1 \varphi_1 \\ \omega_2 \varphi_2 \end{bmatrix} = \begin{bmatrix} \omega_1 a_1 b \\ \omega_2 a_2 b \end{bmatrix} \exp\left(-i\omega_0 t'\right)$$

Where ω_1, ω_2 are complexes

If Φ is a result of calculation as following

$$\Phi = \frac{2}{3}\frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l} b^T \begin{bmatrix} \begin{matrix} a_1^{*T} & 0 \\ 0 & a_2^{*T} \end{matrix} \\ \dfrac{c\sigma\cdot\left(-i\hbar\nabla\right)}{E+mc^2}\begin{bmatrix} a_1^{*T} & 0 \\ 0 & a_2^{*T} \end{bmatrix} \end{bmatrix} \varphi dx'dy'dz'$$

Wher $\frac{2}{3}$ is for the formula after this

b^T is the Transposed matrix of b

And a^{*T} is the Conjugate transpose matrix of a .

With adding former formula then

$$\Phi = \frac{2}{3}\frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l} b^T \begin{bmatrix} \begin{matrix} a_1^{*T} & 0 \\ 0 & a_2^{*T} \end{matrix} \\ \dfrac{c\sigma\cdot\left(-i\hbar\nabla\right)}{E+mc^2}\begin{bmatrix} a_1^{*T} & 0 \\ 0 & a_2^{*T} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \omega_1 a_1 b \\ \omega_2 a_2 b \end{bmatrix} \exp\left(-i\omega_0 t'\right) dx'dy'dz'$$

$$= \frac{2}{3}\frac{1}{l^3}\int_{x=0}^{l}\int_{y=0}^{i}\int_{z}^{l} \begin{bmatrix} b^T \begin{bmatrix} a_1^{*T} a_1 \ , 0 \\ 0\ , a_2^{*T} a_2 \end{bmatrix} b \\ \dfrac{c\sigma\cdot\left(-i\hbar\nabla\right)}{E+mc^2} b^T \begin{bmatrix} a_1^{*T} a_1 \ , 0 \\ 0\ , a_2^{*T} a_2 \end{bmatrix} b \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x} - Et\right)\right) dx'dy'dz'$$

$$= \frac{2}{3}\begin{bmatrix} \frac{3}{2}\,,0 \\ 0\,,\frac{3}{2} \\ \frac{c\sigma\cdot\vec{p}}{E+mc^2}\begin{bmatrix} \frac{3}{2}\,,0 \\ 0\,,\frac{3}{2} \end{bmatrix} \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \frac{c\sigma\cdot\vec{p}}{E+mc^2}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

If ω_3, ω_4 is complexes that's values are as follows

$$\begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix} = \frac{c\sigma\cdot\vec{p}}{E+mc^2}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

then

$$\Phi = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

We differentiate Φ by time t

$$\frac{\partial\Phi}{\partial t} = \frac{\partial}{\partial t}\left(\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)\right) = -\frac{i}{\hbar}E\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

And we seek the gradient of Φ

$$\nabla\Phi = \nabla\left(\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\\\omega_4\end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)\right) = \frac{i}{\hbar}\vec{p}\cdot\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\\\omega_4\end{bmatrix}\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

If σ is spin matrix of Pauli

$$\sigma = \begin{bmatrix}\sigma_x & \sigma_y & \sigma_z\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} & \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} & \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\end{bmatrix}$$

then

$$\left(c\sigma\cdot\vec{p}\right)^2 = c^2\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^2 p_x{}^2 + \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}^2 p_y{}^2 \quad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}^2 p_z{}^2\right)$$

$$= c^2\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}p_x{}^2 + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}p_y{}^2 \quad \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}p_z{}^2\right)$$

$$= c^2 p^2$$

We make a product Φ with matrix $\begin{bmatrix}mc^2 & c\sigma\cdot\left(-i\hbar\nabla\right)\\c\sigma\cdot\left(-i\hbar\nabla\right) & -mc^2\end{bmatrix}$

$$\begin{bmatrix}mc^2 & c\sigma\cdot\left(-i\hbar\nabla\right)\\c\sigma\cdot\left(-i\hbar\nabla\right) & -mc^2\end{bmatrix}\Phi = \begin{bmatrix}mc^2 & c\sigma\cdot\vec{p}\\c\sigma\cdot\vec{p} & -mc^2\end{bmatrix}\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\\\omega_4\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$= \begin{bmatrix}mc^2\begin{bmatrix}\omega_1\\\omega_2\end{bmatrix} + c\sigma\cdot\vec{p}\begin{bmatrix}\omega_3\\\omega_4\end{bmatrix}\\c\sigma\cdot\vec{p}\begin{bmatrix}\omega_1\\\omega_2\end{bmatrix} - mc^2\begin{bmatrix}\omega_3\\\omega_4\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=\begin{bmatrix} mc^2\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}+c\sigma\cdot\vec{p}\,\dfrac{c\sigma\cdot\vec{p}}{E+mc^2}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \\ c\sigma\cdot\vec{p}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}-mc^2\,\dfrac{c\sigma\cdot\vec{p}}{E+mc^2}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=\begin{bmatrix} \left(mc^2+\dfrac{c^2p^2}{E+mc^2}\right)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \\ \left(c\sigma\cdot\vec{p}\right)\left(1-\dfrac{mc^2}{E+mc^2}\right)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=\begin{bmatrix} \left(\dfrac{Emc^2+m^2c^4+c^2p^2}{E+mc^2}\right)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \\ \left(c\sigma\cdot\vec{p}\right)\left(\dfrac{E}{E+mc^2}\right)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=\begin{bmatrix} \left(\dfrac{Emc^2+E^2}{E+mc^2}\right)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \\ \dfrac{\left(c\sigma\cdot\vec{p}\right)E}{E+mc^2}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=\begin{bmatrix} E\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \\ E\begin{bmatrix}\omega_3\\ \omega_4\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)=E\begin{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}\end{bmatrix}\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x}-Et\right)\right)$$

$$=i\hbar\frac{\partial\Phi}{\partial t}$$

Therefore

$$i\hbar\frac{\partial\Phi}{\partial t}=\begin{bmatrix} mc^2 & c\sigma\cdot\vec{p} \\ c\sigma\cdot\vec{p} & -mc^2\end{bmatrix}\Phi$$

This equation is consistent with the Dirac equation.

References

Mathematical formulas

$$\cos(A) - \sin(A) = -\sqrt{2}\,\sin\left(A - \frac{\pi}{4}\right)$$

$$\sin(A) + \cos(A) = \sqrt{2}\,\sin\left(A + \frac{\pi}{4}\right)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\exp(i\theta) = \cos(\theta) + i(\sin(\theta))$$